The symbol $g(f(x))$, read "g of f of x," is a **composition** of the two functions f and g. The composition $g(f(x))$ gives the final outcome when an x-value is substituted into the inner function f, and its value $f(x)$ is then substituted into the outer function g.

You have actually been composing functions when you transformed graphs using two or more steps. The function $3f(x) - 1$ is obtained by first stretching a function $f(x)$ by a factor of three to get a new image, and then subtracting 1 from these new y-values to slide the graph down one unit. (Remember to perform stretches before you do any vertical translations.)

Example 2

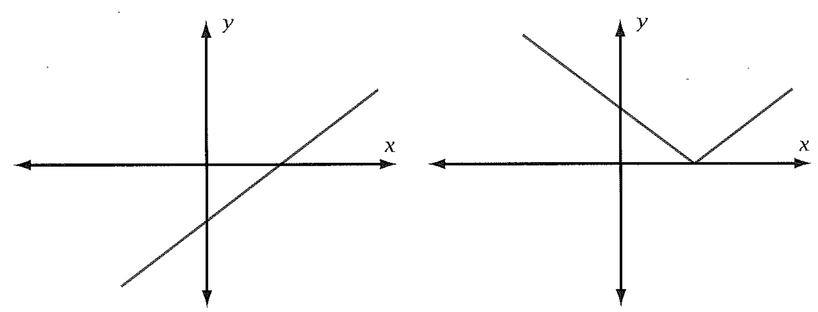
Consider the line pictured on the left below as an inner function, perhaps

$$f(x) = \frac{3x}{4} - 3.$$

Suppose g is the absolute value function. Then $g(f(x))$ will be the absolute value of the inner linear function $f(x)$. What will $g(f(x))$ look like?

Solution

The solution is the composition graph on the right. Take a minute or so to make sure you understand why this graph shows the solution.



Example 3

Suppose $g(x) = 3x - 7$ and $f(x) = \frac{1}{x}$.

Determine the value for each function composition.

a. $f(g(4))$ **b.** $f(g(x))$ **c.** $D_{f(g(x))}$ **d.** $g(f(4))$

Solution

a. First, $g(4) = 3(4) - 7 = 5$. Therefore, $f(g(4)) = f(5) = \frac{1}{5}$.

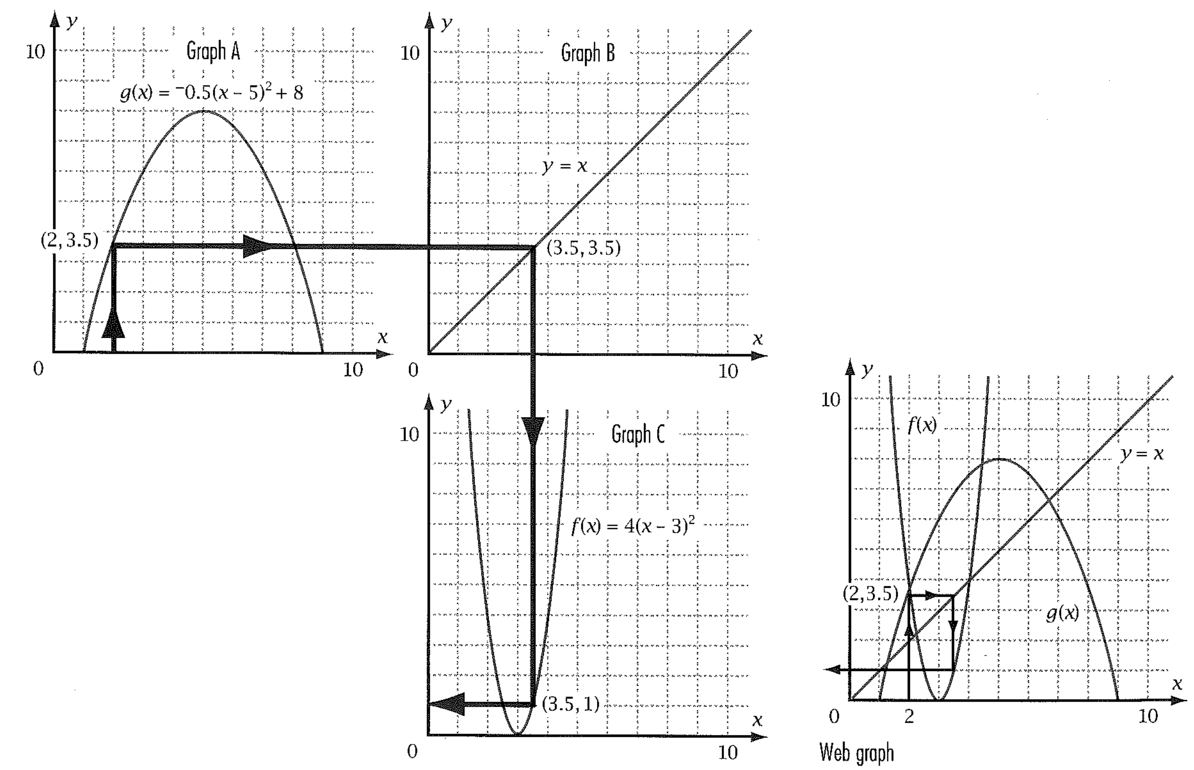
b. By definition, $g(x) = 3x - 7$. Therefore, $f(g(x)) = f(3x - 7) = \frac{1}{3x - 7}$.

c. The domain required will contain all real numbers except $\frac{7}{3}$, because you cannot allow the denominator of $\frac{1}{3x - 7}$ to be zero.

d. First, $f(4) = \frac{1}{4}$. Therefore, $g\left(\frac{1}{4}\right) = 3\left(\frac{1}{4}\right) - 7 = {}^-6\frac{1}{4}$.

One way to visualize what is happening when you do a composition of functions is to use a three-part graphing procedure. Graph A shows $g(x) = {}^-0.5(x - 5)^2 + 8$, Graph B shows the line $y = x$, and Graph C shows $f(x) = 4(x - 3)^2$.

- Choose an x-value. In this example, an x-value of 2 has been chosen.
- Evaluate $g(2)$ by drawing a vertical line from the x-axis to the function on Graph A.
- Then draw a horizontal line from that point to the line $y = x$ on Graph B. The point where this line intersects the graph has x- and y-coordinates that are the same.
- Now draw a vertical line from this point so that it intersects the graph of the parabola in Graph C. The y-value of this point gives the same result as evaluating f at the y-value of the original function.
- Draw a horizontal line from the intersection point to the y-axis.
- The y-value is $f(g(2))$, or 1.



Use the method demonstrated to find $f(g(1.5))$ and $f(g(8.5))$. Trace the path for each composition and estimate the answer. Confirm the result by first calculating $g(x)$ and then substituting this answer into $f(x)$.

This procedure can be shortened by placing each graph on the same axes. When you do this, the graph is called a **web graph**, as shown above. The path then moves from the

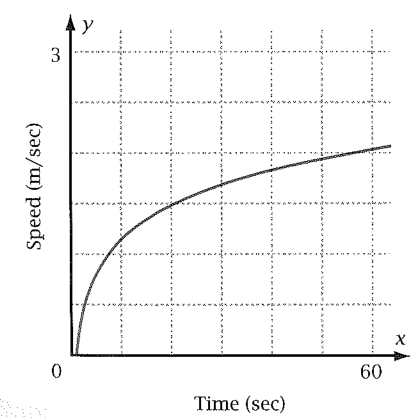
x-axis to $g(x)$ to the line $y = x$ to $f(x)$ to the y-axis. The order is very important. A modification of this procedure allows you to graphically show $f(f(x))$. In the following investigation, you will explore this process for a specific function.

Investigation 5.9.1
$f(f(f(f(\ldots f(x)\ldots))))$

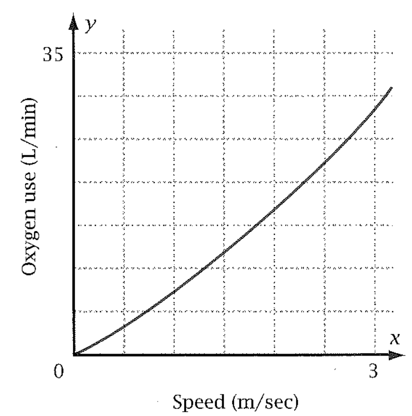
You will need a worksheet with graphs of $y = ax(1 - x)$ for various values of a. Begin with an x-value of 0.3. Use a graphical method similar to the one described above to find $f(f(f(f(\ldots f(x)\ldots))))$. Carefully draw each web graph. Repeat the graphical steps enough times to be able to predict what is going to happen. Did everyone in your group draw the same types of graphs? Describe what happened in each case.

Problem Set 5.9

1. Graph A shows a swimmer's speed as a function of time. Graph B shows the swimmer's oxygen consumption as a function of her speed. Time is measured in seconds, speed in meters per second, and oxygen consumption in liters per minute.



Graph A

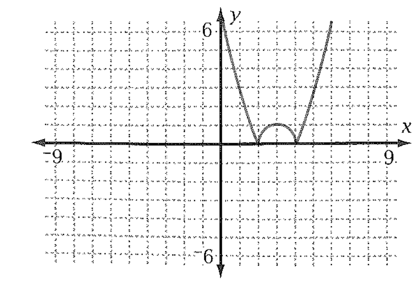


Graph B

a. Use the graphs to find the swimmer's oxygen consumption after 20 sec of swimming.

b. Sketch the graphs on your paper and draw segments on both graphs that verify your thinking.

c. How many seconds have elapsed if the swimmer's oxygen consumption is 15 L/min?



2. a. Write the equation in $y=$ form for the graph pictured at the right.

b. Invent two functions f and g so that the figure is the graph of $f(g(x))$.

3. A, B, and C are thermometers with different linear scales. When A reads 12 and 36, B reads 13 and 29, respectively. When B reads 20 and 32, C reads 57 and 84, respectively.

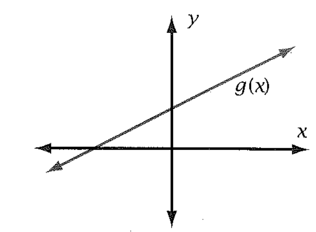
a. Sketch a separate graph for each function. Label the axes.

b. If A reads 12, what does C read?

c. Write a function with B depending on A.

d. Write a function with C depending on B.

e. Write a function with C depending on A.



4. A graph of the function is shown at the right. Draw a graph of each related function, $h(x)$, given below.

a. $h(x) = \sqrt{g(x)}$

b. $h(x) = |g(x)|$

c. $h(x) = g(x)^2$

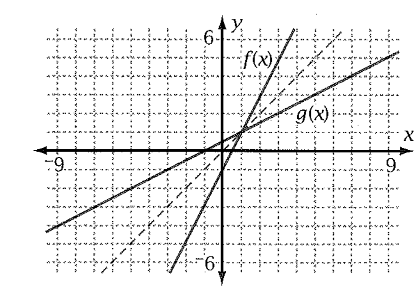
5. Suppose $g = \{(1, 2), (^-2, 4), (5, 5), (6, ^-2)\}$ and $f = \{(0, ^-2), (4, 1), (3, 5), (5, 0)\}$.

a. Find $g(f(4))$. **b.** Find $f(g(^-2))$.

6. Suppose $g = \{(1, 2), (^-2, 4), (5, 5), (6, ^-2)\}$ and $f = \{(2, 1), (4, ^-2), (5, 5), (^-2, 6)\}$.

a. Find $g(f(2))$. **b.** Find $f(g(6))$.

c. Select any number from the domain of either g or f and find its composite value by using the two functions. Describe what is happening.



7. The two graphs pictured at the right are $f(x) = 2x - 1$ and $g(x) = \frac{1}{2}x + \frac{1}{2}$. Begin by making an accurate copy of the graph. Solve each problem both graphically and numerically.

a. Start with $x = 2$, and find $g(f(2))$.

b. Start with $x = ^-1$, and find $f(g(^-1))$.

c. Pick your own starting x-value in the domain of f, and find $g(f(x))$.

d. Pick your own starting x-value in the domain of g, and find $f(g(x))$.

e. Carefully describe what is happening in these compositions.

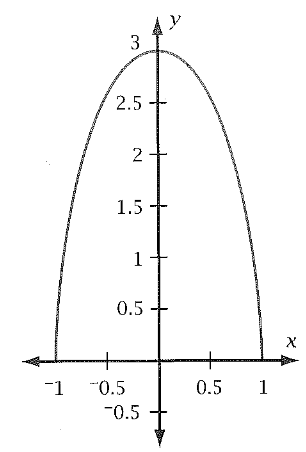
8. Suppose $f(x) = ^-x^2 + 2x + 3$ and $g(x) = (x - 2)^2$. Find each value below, both graphically and algebraically.

a. $f(g(3))$ **b.** $f(g(2))$ **c.** $g(f(0.5))$ **d.** $g(f(1))$

NOTE 5F

9. Your calculator can create graphical compositions like those in Investigation 5.9.1. (See **Calculator Note 5F** for specific instructions.) Use your calculator to verify your conclusions from the investigation. Determine the value that each function appears to approach in the long run as you continue the process.

10. Aaron and Davis are both studying the graph shown at the right. They need to write the equation that will produce this graph.



"This is impossible!" Aaron exclaims. "How are we supposed to know if the parent function is a parabola or a semicircle? If we don't know the parent function, there is no way to write the equation."

"Don't panic yet," Davis replies. "I am sure we can determine its parent function if we study the graph carefully."

Who is correct? Explain completely and, if possible, write the equation of the graph.

Section 5.10

Chapter Review

I cannot judge my work while I am doing it. I have to do as painters do, stand back and view it from a distance, but not too great a distance. How great? Guess.
—Blaise Pascal

In the first part of this chapter you learned how linear equations and arithmetic sequences are related. The concept of a function was also introduced. You explored several families of functions—linear, quadratic, square root, and absolute value, and you learned how to move and stretch these functions. For example, in the equation $y = 3((x - 1)/2)^2 + 4$, the parent function, $y = x^2$, has been stretched by a factor of three vertically, stretched by a factor of two horizontally, moved to the right one unit, and moved up four units. When applying transformations to a parent function, remember to apply the stretches before moving the graph out of its original position. The same rules of transformation apply to all functions.

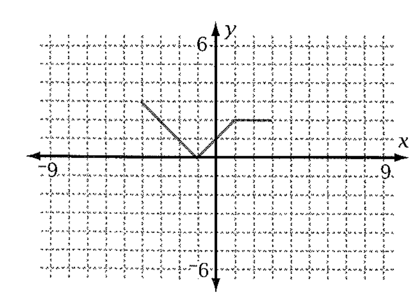
Finally, you looked at the composition of functions. Many times solving a problem involves two or more related functions. You can find the value for a function composition by using algebraic or numeric methods or by graphing. Be sure that you understand how to use the different methods.

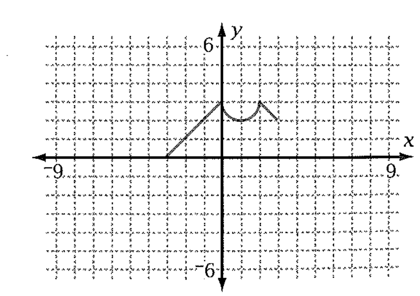
Problem Set 5.10

1. Sketch a graph showing the relationship between the number of pops per second and the time since you plugged in the popcorn popper. Describe in words what your graph is showing.

2. If $f(x) = {}^{-}2x + 7$, $g(x) = x^2 - 2$, and $h(x) = (x + 1)^2$, find each value.
   a. $f(g(3))$ **b.** $g(h({}^{-}2))$ **c.** $h(f({}^{-}1))$
   d. $f(g(x))$ **e.** $h(f(x))$ **f.** $g(f(x))$

3. For each function pictured below, draw the indicated transformation.
   a. $f(x) - 3$ **b.** $f(x - 3)$

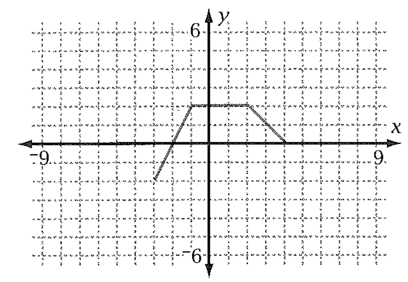




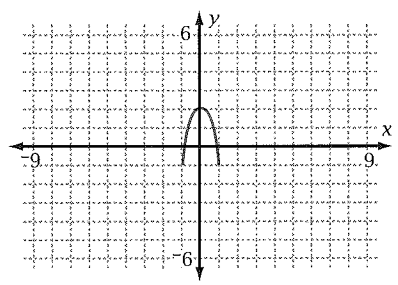
4. Describe the correct order for performing the combination of transformations on each function $f(x)$ given below.
   a. $f(x + 2) - 3$
   b. $^{-}f\left(\frac{x}{2}\right) + 1$
   c. $2f\left(\frac{x-1}{0.5}\right) + 3$

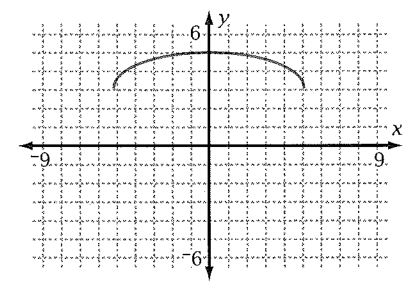
5. Solve for y.
   a. $2x - 3y = 6$
   b. $(y + 1)^2 - 3 = x$
   c. $\sqrt{1 - y^2} + 2 = x$

6. The graph of $y = f(x)$ is given. Draw each transformation, or combination of transformations, of this function on a separate axis.
   a. $f(x) - 2$
   b. $f(x - 2) + 1$
   c. $^{-}f(x)$
   d. $2f(x + 1) - 3$
   e. $f(^{-}x) + 1$
   f. $f\left(\frac{x}{2}\right) - 2$
   g. $^{-}f(x - 3) + 1$
   h. $^{-}2f\left(\frac{x-1}{1.5}\right) - 2$



7. For each graph, name the parent function and write an equation of the graph.

a.


b.


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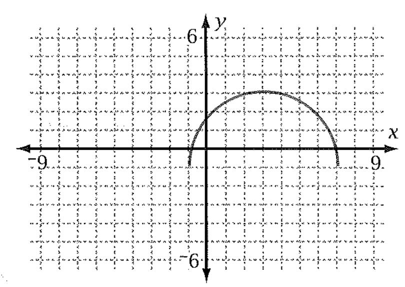

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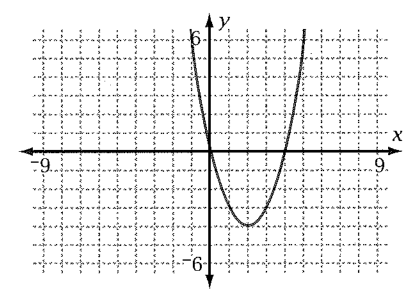
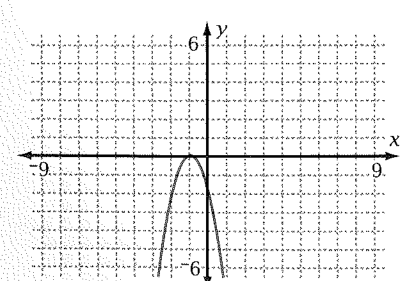


e.


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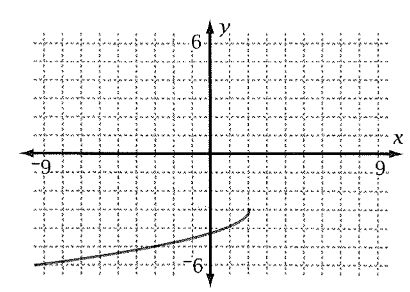
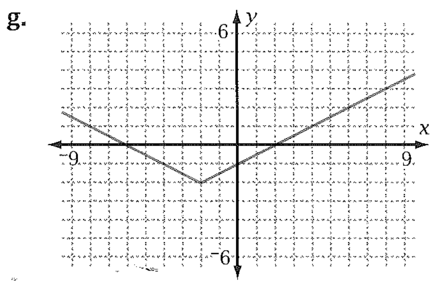


g.


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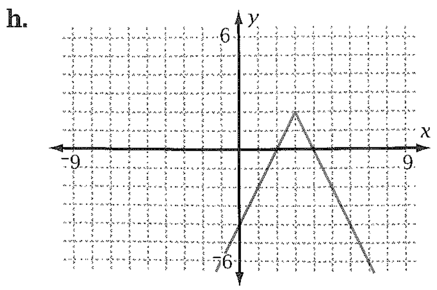

Project

Melting Ice

Algebra student Will Melt carefully crafted a device from a wire coat hanger and a rubber band. He placed this device on a scale and attached an ice cube to the rubber band. He then carefully read the mass every 10 minutes for 100 minutes, at which time the ice cube dropped off the rubber band. Because melting occurs on the surface of the ice and surface area is measured in square centimeters, he was sure the relationship would be quadratic. The data he collected is in the table below. Time is recorded in minutes and mass is recorded in grams.

Time (min)	0	10	20	30	40	50	60	70	80	90	100
Mass (g)	52.4	51.9	50.8	49.4	47.9	46.4	45	43.3	42.1	40.6	39.2

Much to his dismay, he discovered the data was quite linear. Plot the data and draw the least-squares line. (Be sure to label the scale and units in each graph you create in this project.)

Upon reflection, Will recalled that he needed to subtract the mass of the hanger and the rubber band, which totaled 34.3 grams. Subtract this from the original y-values and plot the points with the new y-values on the same graph. Find the equation that fits this new data.

Will then decided that, because little significant melting happened in the first 10 minutes, he would subtract 10 from each of the x-values. Graph the data and the equation for this set of data.

Things still weren't working out as he thought they should, so he converted grams to ounces by dividing each y-value by 28.35. He then repeated his analysis. Add this graph and equation to your report.

Still not satisfied, Will decided to convert the time to seconds, so he multiplied all the x-values by 60. He thought again about the coat hanger and wondered if he should subtract its weight first and then change to ounces, or change to ounces and then subtract the weight. Try these two to determine which is correct.

(continued on next page)

Even after all this, he was not content, so he thought he would check the mass of the melted water. He made all the y-values negative and added the mass of the original cube.

Write a summary of what Will, and you, learned at each step of this analysis. Explain how each step relates to the transformation of functions.

Assessing What You've Learned—Constructing Test Questions

Have you ever wondered how your teacher decides what to put on a test or which questions to ask in order to assess what students have learned? One thing that teachers may consider is the amount of time spent on each topic. Other considerations may include the importance of a topic, its application, or perhaps the special interest that it generated during class discussions. Writing good test items is not an easy task, but it is another way for you to assess what you have learned.

Think about the investigations and problems you have completed in this chapter. By yourself or with your group, write at least five problems that you think could be used to assess what a student should have learned. Some of the questions you create may pertain to specific skills you have acquired, but try to write a few that require a student to apply problem-solving strategies. Your teacher may even choose some of your problems to include on the next test.

Continue using the other methods that have been described to assess what you've learned. You might not be using them all, but it's a good idea to assess yourself using a variety of methods.

Organizing Your Notebook

- By taking just a few minutes each day, you can keep your notebook in shape and avoid the hassle of having to locate and organize your work when you complete a chapter.

Keeping a Journal

- Look in Problem Sets 5.7 and 5.8 for journal prompts.

Portfolio

- Have you done any work in this chapter that you would like to include in your portfolio? If so, document it, and put it in.

Performance Assessment

- Demonstrate to someone that you know how each of the transformations introduced in this chapter will affect the graph of a function.

Open-ended Investigations

- Choose one of the Take Another Look investigations or a project, or design your own investigation. Share your results with a classmate, your group, or your teacher.

6

Parametric Equations and Trigonometry

Not only do these parachutists require a great deal of teamwork to make this maneuver, but they must also have an understanding of distance, rate, and time relationships. These parachutists are falling at a rate of 120 miles per hour, thousands of feet high in the sky. They will break their formation and open their parachutes when they are about 2,000 feet from the ground. You can use your parametric equations to model the relationships among distance, rate, and time.